Parametric Quantum Resonances for Bose-Einstein Condensates

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We generalize recent work on parametric resonances for nonlinear Schrödinger (NLS) type equations to the case of three dimensional Bose-Einstein condensates at zero temperatures. We show the possibility of such resonances in the three-dimensional case, using a moment method and numerical simulations.

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I. INTRODUCTION

There has been a blossoming of literature on the features of systems exhibiting Bose-Einstein condensation (BEC), triggered by its recent experimental realization [1]. Intially experiments with 10^3 to 10^6 atoms of rubidium or sodium (later experiments have used lithium and eventually (spin-polarized) hydrogen), in harmonic or cigar shaped traps have demonstrated condensation to a "pseudo-macroscopic" level of occupancy of the ground state for nK temperatures. Time of flight measurements, velocity distributions as well as spatial profiles have convincingly supported the physical picture of an abrupt transition in the behavior of the Bose gas, which has been interpreted as the signature of BEC.

Following these experiments, many theoretical studies were launched to characterize different aspects of Bose condensates such as hydrodynamic modes [2], collective excitations [3], the behavior of ideal quantum fluids [4,5], the fraction of noncondensate vs. condensate atoms [6,7], or the generation and stability of vortices [8]. In turn, experimental studies have progressed to address some of the theoretical predictions [9] and open up new questions.

Here, we concern ourselves with one aspect of these quantum fluids, namely parametric driving. For the purpose of this report, we will restrict ourselves to the framework of the mean-field or Hartree-Fock approximation. This approximation is rigorously justifiable only at $T = 0$ but it is expected [6] that the contribution of the noncondensate to the density is quite small. It is well-known that at this mean-field level the condensate wavefunction is governed by the Gross-Pitaevskii (GP) [10] equation. An issue addressed after the original experiments achieving the condensation was the study of collective excitations [3,11]. In these papers these excitations were induced by a harmonic trap weakly modulated in time with appropriate types of symmetry. More recently, it was demonstrated that extended parametric resonances can occur [12] in a two-dimensional (2d) NLS equation with a harmonic trap. This result may or may not (for reasons to be explained below) be relevant for two dimensional studies of Bose gases. However, this naturally raises the question of whether a similar result can be de-

duced for the $3d$ case which is certainly of direct relevance to experimental studies.

The main question we will address is whether weak harmonic modulation of trapped $3d$ condensates can cause an anomalously large response in their wavefunction. Our answer, which will be in the affirmative, will be motivated by mathematical analysis using a moment method and verified by numerical simulation. We will briefly discuss the implications of these results and the suggestion of relevant experiments.

II. MOMENT METHODS

Considering a spherical trap, the dimensionless GP equation for the dynamics of the BEC condensate is [13]

$$iu_t = -\frac{1}{\zeta^4}\nabla^2 u + \left(\lambda(t)r^2 + \nu|u|^2\right)u. \tag{1}$$

Here, the subscript t denotes time derivative and $\zeta = (8\pi N|a|/a_\perp)^{1/5}$ is a dimensionless parameter arising from the number N of particles, the s-wave scattering length a and from $a_\perp$ characterizing the strength of the trap (see, Ref. [13]). In Eq.(1) $\lambda(t)$ is a dimensionless function allowing for time dependence of the trap and $\nu = \text{sign}(a)$, generalizes the equation to describe attractive $(a < 0)$ as well as repulsive $(a > 0)$ interactions. Since, we restrict ourselves to spherical symmetry we only include the radial contribution in the Laplace operator. Although we are mainly interested in the full three-dimensional $(3d)$ case we will in general consider the d-dimensional version of Eq.(1) so that

$$\nabla^2 = \frac{1}{r^{d-1}}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial}{\partial r}\right). \tag{2}$$

Similarly to Ref. [12], we define the following quantities

$$I_{2,a}^{(d)} = \int_0^\infty r^a |u|^2 r^{d-1} dr, \tag{3}$$

$$I_{3,a}^{(d)} = i \int_0^\infty r^a (uu_r^\star - c.c.) r^{d-1} dr, \tag{4}$$

$$I_{4,a}^{(d)} = \int_0^\infty r^a \left|\frac{\partial u}{\partial r}\right|^2 r^{d-1} dr, \tag{5}$$

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$$I_{5,a}^{(d)} = \int_0^\infty r^a |u|^4 r^{d-1} dr, \tag{6}$$

where (d) indexes the dimension. This type of nonlinear Schrödinger (NLS) equation (1) has two conserved quantities: as a result of the phase invariance the *norm* corresponding to $I_{2,0}^{(d)}$ is conserved in any dimension d. Also, since Eq.(1) is a Hamiltonian system arising from

$$H = \int_0^\infty \left[\zeta^{-4} |\nabla u|^2 + \frac{\nu}{2} |u|^4 + \lambda(t) r^2 |u|^2 \right] r^{d-1} dr \tag{7}$$

this quantity is conserved. It is useful to note that the Hamiltonian (or more appropriately *energy functional*), H, can be expressed in terms of the moments Eqs.(3)-(6)

$$H = \zeta^{-4} I_{4,0}^{(d)} + \frac{\nu}{2} I_{5,0}^{(d)} + \lambda(t) I_{2,2}^{(d)}. \tag{8}$$

The relevance of the moments Eqs.(3)-(6), is based on their time evolution and in the following we will derive the relations governing this dynamics. The physical rationale behind such an approach lies in the fact that the resulting equations can yield predictive diagnostics for the dynamics of the BEC. In particular, $I_{2,2}^{(d)}$ will essentially yield the width of the spatial profile of the wavefunction. If a condensation phenomenon is to take place even when starting from a spatially uniform distribution (at temperatures $T < T_c$), the width must evolve towards a constant non-zero value.

Using Eq.(1) and its complex conjugate one can derive

$$\dot{I}_{2,a}^{(d)} = \zeta^{-4} a I_{3,a-1}^{(d)}, \tag{9}$$

$$\dot{I}_{3,a}^{(d)} = -4\lambda(t) I_{2,a+1}^{(d)} + 4a\zeta^{-4} I_{4,a-1}^{(d)}$$
$$- (a+d-1)(a-1)(a-3+d)\zeta^{-4} I_{2,a-3}^{(d)} \tag{10}$$
$$+ \nu(a+d-1) I_{5,a-1}^{(d)}.$$

In deriving these we assume u to vanish as $r \to \infty$.

Unfortunately, it is in general impossible to close this hierarchy of equations because the time derivative couples to the next order e.g. $\dot{I}_{2,a}^{(d)}$ couples to $I_{3,a-1}^{(d)}$ and $\dot{I}_{3,a}^{(d)}$ couples to $I_{5,a-1}^{(d)}$ and $I_{4,a-1}^{(d)}$, and so on. However, combining Eqs. (9) and (10) provides some insight

$$\ddot{I}_{2,a}^{(d)} = \zeta^{-4} a \left[-4\lambda(t) I_{2,a}^{(d)} + 4(a-1)\zeta^{-4} I_{4,a-2}^{(d)} \right.$$
$$- (a+d-2)(a-2)(a-4+d)\zeta^{-4} I_{2,a-4}^{(d)} \tag{11}$$
$$\left. + \nu(a+d-1) I_{5,a-2}^{(d)} \right].$$

First, this confirms that the norm $I_{2,0}^{(d)}$ is conserved in any dimension. Secondly, this relation clearly suggests $a = 2$ as good choice since the term involving $I_{2,a-4}^{(d)}$ then vanishes irrespective of dimension. Also, this choice allows the use of Eq.(8) to reduce the expression (12) to

$$\ddot{I}_{2,2}^{(d)} = 8\zeta^{-4} H - 16\zeta^{-4}\lambda(t) I_{2,2}^{(d)} + 2\nu(d-2) I_{5,0}^{(d)}, \tag{12}$$

which corresponds to the relation commonly referred to as the *virial theorem* [14] for the nonlinear Schödinger without a trap, $\lambda(t) \equiv 0$.

Clearly, the $2d$ case is special as the structure of Eq.(12) is such that a closed time evolution can be prescribed. In addition, for time dependent modulation of the trap amplitude, we find (as was observed for this problem in Ref. [15] and again in Ref. [12]) a Hill type equation which establishes parametric resonances for the behavior of the width (or amplitude) of the wavefunction. Important as this conclusion about the $2d$ behavior may be for general NLS-GP equations, it is not clear that it is relevant to BEC. Since BEC is not possible in spatial dimensions less that three ($d < 3$) [5] where a Kosterlitz-Thouless topological transition seems to be occurring instead [5,16], the applicability of the GP equation for $d < 3$ is controversial. Our search for condensate instabilities is therefore most compelling in three dimensions where no such reservations exist.

Although Eq.(12) is not closed in $3d$, it is easily seen that the following inequalities hold for $d \geq 2$

$$\ddot{I}_{2,2}^{(d)} + 16\zeta^{-4}\lambda(t) I_{2,2}^{(d)} \leq 8\zeta^{-4} H \quad \text{for} \quad \nu < 0, \tag{13}$$

$$\ddot{I}_{2,2}^{(d)} + 16\zeta^{-4}\lambda(t) I_{2,2}^{(d)} \geq 8\zeta^{-4} H \quad \text{for} \quad \nu > 0, \tag{14}$$

where the equality applies to the two-dimensional case only (and in fact for the noninteracting $3d$ case $\nu = 0$). The value of these inequalities lies in the predictions about the $3d$ case [17]. Since we can resolve, or at least very well characterize, the $2d$ behavior, we are now able to extend this to quantitative predictions about of the $3d$ behavior. For the attractive case $\nu < 0$ the possibility of collapse occurs as $I_{2,2}^{(d)}$ can become zero in finite time. In $2d$ and due to Eq.(14) also in $3d$, a sufficient condition for collapse is $H < 0$, although, depending on the initial configurations, collapse can be achieved even for $H > 0$. A more complete discussion is given in Ref. [18]. For the more realistic case (in BEC contexts) of repulsive interaction $\nu > 0$ we see for example that the parametric resonances that were demonstrated in the two-dimensional case [12] will also be present in the three-dimensional case. So, for instance, for $d = 3$ and $\lambda(t) = \lambda_0^2(1 + \epsilon \cos(\omega t))$ will exhibit resonance around $\lambda_0 = n\omega/2$, $n = 1, 2, 3..$ where the extent of the first resonance is determined by the inequality $|1 - \omega^2/(4\lambda_0^2)| < \epsilon/2$ [19]. However, additional resonances may be possible in $3d$ due to the nonlinear driving resulting from the repulsive interaction. Some understanding of the influence of the last term in Eq.(12) on the dynamics can be gained by assuming that the wavefunction can be approximated as
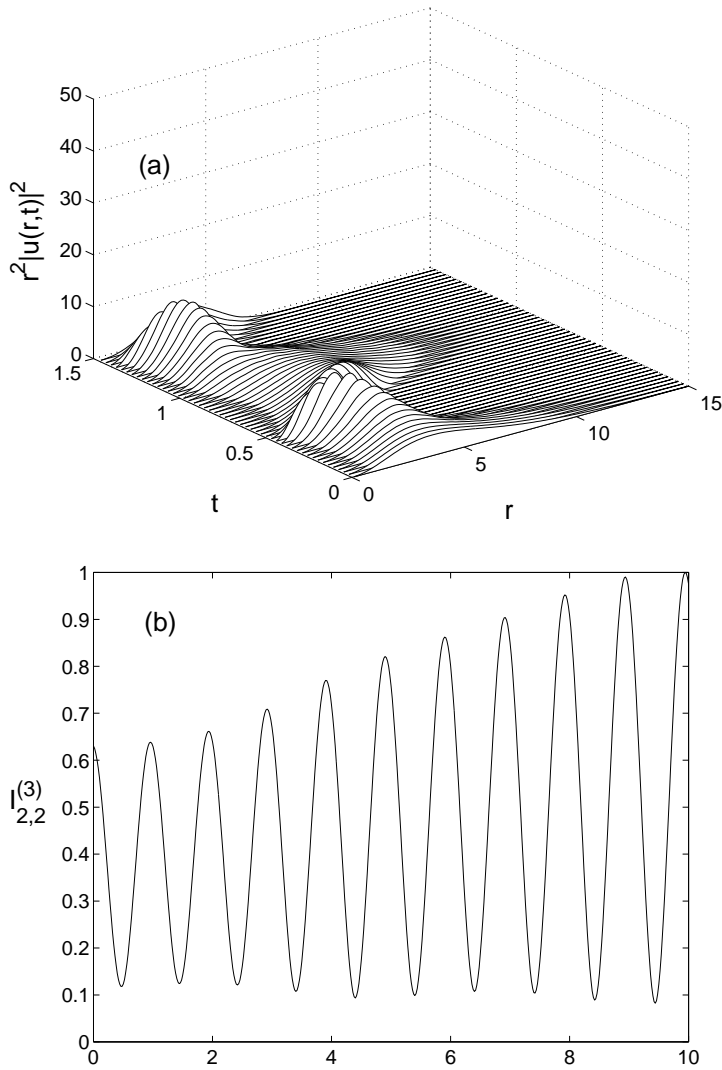
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FIG. 1. (a) evolution (actually $r^2|u(r,t)|^2|$ is plotted for clarity) of the condensate wavefunction both in space (r) and in time (t) for part of the domain (close to $r = 0$) and (b) the evolution if of $I_{2,2}^{(3)}$. Parameters are $\lambda_0 = 1$, $\omega = 1$, (i.e. second resonance) and $\epsilon = 0.05$.

$$u = \sqrt{\frac{I_{2,0}^{(3)}}{\sigma_1}} B^{-3/2} \psi(r/B), \qquad (15)$$

where $\sigma_1 = \int_0^\infty r^2 \psi(r) dr$ is a shape-dependent constant. Thus, assuming adiabatically the wavefunction does not alter its shape ψ significantly as a result of the dynamics, u as defined in Eq. (15) automatically satisfies the norm conservation. Utilizing this in Eq.(12) yields

$$\frac{d^2 B^2}{dt^2} - \lambda(t)B^2 = Q_1 + \nu Q_2 B^{-3}, \qquad (16)$$

where Q_1 and Q_2 are constants determined by the shape ψ and the initial value of B. Clearly, the last term in Eq.(16) will only influence the dynamics when B becomes small. In the repulsive $\nu > 0$ case however B will generally not become small since there is no collapse. This simple analysis suggests that the parametric forcing of the experimentally realizable $3d$ case will result in a resonance picture analogous to that previously reported for the $2d$ problem. Our numerical simulations of the full Eq.(1) with $\lambda(t)$ as given above verifies the validity of this prediction. A typical example for $\lambda_0 = 1$, $\omega = 1$, (i.e. $n = 2$) and $\epsilon = 0.05$ is given in Fig.1. The response of the wavefunction (whose initial condition had $\max_x |\psi(x, t = 0)|^2 = 1$), corresponding to the parametric resonance, can be observed directly from the wave function Fig. 1(a) but even more clearly in the time evolution of $I_{2,2}^{(3)}$, as shown in Fig. 1(b).

III. CONCLUSION AND FUTURE CHALLENGES

In this paper, we have presented and extended the formalism of the moment method, used in Refs. [12] and [20] for the $2d$ GP equations, to the more relevant $3d$ case. We have commented on the special nature of the two-dimensional problem where the moment equations form a closed set of equations. We have also added a note of caution in considering the results of the GP analysis for $d < 3$. It might well be that, analogous to mean-field analysis in statistical physics systems, the "critical dimension" for this system is, indeed, $d_c = 3$ and for lower dimensionalities the predictions of the mean-field theory are unreliable. A satisfactory self-consistent first principles description of an interacting boson gas for $d < 3$ presents a very challenging theoretical problem, and it remains an unresolved issue whether a transition is present (and if it is, what is its nature).

On the other hand, we have used the moment methods and have derived results for GP functionals in all dimensions of physical interest. Considering, in particular, the $3d$ case, where the validity of the GP approximation is clear, we have obtained a non-closed set of equations for the moments of the wavefunction. We have demonstrated that for a parametric time-disturbance of the trap amplitude, parametric resonances are possible. To date the experiments that have used parametric modulation have not observed such phenomena. These experiments have been performed in cigar-shaped traps (where the analysis is considerably more complicated even in the $2d$ problem [12]). No fine tuning of frequencies and amplitudes was explored since the aim of the studies was to excite collective modes rather than to observe parametric resonances. Hence, we propose an experiment in a spherical trap using weak harmonic modulation of the condensate. Given the current experimental advances (see e.g. Ref. [21] for a review), such an experiment seems feasible. Such a study would, apart from the validation of the theoretical prediction, also explore how such resonances might

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destabilize the condensate.

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